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The Hydrodynamic Limit of a One-Dimensional Nearest Neighbor Gradient System

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We study the hydrodynamic behavior of a one-dimensional nearest neighbor gradient system with respect to a positive convex potential Φ. In the hydrodynamic limit the density distribution is shown to evolve according to the nonlinear diffusion equation $\partial\rho_t(q)/\partial t = (\partial^2/\partial q^2)\{\mathbf{F}([1/\rho_t(q)])\}$, with $\mathbf{F} = -\Phi'$.

KEY WORDS: Nearest neighbor gradient system; hydrodynamic limit; nonlinear diffusion.

1. INTRODUCTION

With the aim of understanding how hydrodynamic behavior arises from molecular dynamics, several models have been studied rigorously, but only a few with deterministic dynamics (for a survey see Ref. 1, particularly Section 2).

In this paper we treat the hydrodynamic limit of a one-dimensional nearest neighbor gradient system.

The gradient system is the time evolution of a configuration of particles located at $x_i \in \mathbf{R}^d$ ($i \in \mathbf{Z}$), given by the system of equations

$$\frac{dx_i}{dt} = \sum_{j \neq i} \mathbf{F}(x_i - x_j); \qquad i \in \mathbf{Z}$$

where the force $\mathbf{F}$ is the negative gradient of a symmetric potential $\mathbf{\Phi}$.

One gets the gradient system if, in the classical Newtonian dynamics,

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the second derivative with respect to time is replaced by the first one. It is also related to the system of stochastic differential equations

$$dx_i = \sum_{j \neq i} \mathbf{F}(x_i - x_j)\, dt + \beta^{-1}\, dw_i(t); \qquad i \in \mathbf{Z}$$

with independent Wiener processes w_i $(i \in \mathbf{Z})$ and inverse temperature $\beta > 0$. For a discussion see Lang.[2]

The gradient system is easier to handle than the Newtonian dynamics. Lang[2] and Zessin[3] have studied the spatially homogeneous case, especially the equilibrium distributions and convergence to them.

Our model is the finite, one-dimensional, nearest neighbor gradient system, given by the system of equations:

$$\frac{dx_i}{dt} = \sum_{j:\,|j-i|=1} \mathbf{F}(x_i - x_j) = -\mathbf{F}(x_{i+1} - x_i) + \mathbf{F}(x_i - x_{i-1}); \qquad 1 \leqslant i \leqslant N$$

with $x_i < x_{i+1}$ for $1 \leqslant i \leqslant N-1$. The force $\mathbf{F} = -\mathbf{\Phi}'$ is derived from a symmetric, positive, convex potential $\mathbf{\Phi}$.

As we study the finite particle case, we tacitly take the corresponding term for 0 if no particle to the right (resp. left) exists.

The equation has the same form as Spitzer's model of unbounded spins.[4] There x_i represents the spin of a particle located at the lattice site $i \in \mathbf{Z}$.

The hydrodynamic limit of Spitzer's model has been derived by Fritz[5] and Presutti and Scacciatelli,[6] but because of the different interpretation the assumptions for the gradient system are different. One difference consists in the interaction, which is attractive in Spitzer's model, but repulsive in the gradient system.

Hydrodynamic behavior first requires the transition from microscopic to macroscopic scales of space and time with their ratio tending to 0. This kind of limit is called the hydrodynamic limit. It has to be distinguished from cases where in addition the dynamics is rescaled, too, such as the Boltzmann–Grad limit.

For the gradient system the diffusive scaling is appropriate, i.e., for $\varepsilon > 0$ we set

$$q_i(t) = \varepsilon x_i(\varepsilon^{-2} t)$$

which evolves according to the system of equation

$$\frac{dq_i}{dt} = \varepsilon^{-1} \left\{ -\mathbf{F}\left(\frac{q_{i+1} - q_i}{\varepsilon}\right) + \mathbf{F}\left(\frac{q_i - q_{i-1}}{\varepsilon}\right) \right\}; \qquad 1 \leqslant i \leqslant N \qquad (1.1)$$

Another difference from Spitzer's model concerns this scaling, which has to be applied to the different quantities representing the location of the particles.

Since the number of particles in bounded intervals is of finite order on the microscopic level, it diverges of the order ε^{-1} on the macroscopic level. Likewise, we take N of the order ε^{-1}.

A further general feature of hydrodynamic behavior is the significance of the conserved quantities and the formation of local equilibrium.

As (1.1) indicates, on the macroscopic level the right-hand side of the dynamical equations diverges as $\varepsilon \downarrow 0$. In the time derivatives of the distributions of the conserved quantities, however, at least formally the divergence cancels. Thus, one would suppose a smooth macroscopic behavior of them under suitable conditions.

Furthermore, the conserved quantities parametrize the equilibrium distributions. Approach to equilibrium for large microscopic times in the spatially homogeneous case suggests the validity of local equilibrium for strictly positive macroscopic times. This means that in microscopic neighborhoods of macroscopic points the distribution of the configuration is approximately in equilibrium with the corresponding parameters varying with space and time. A precise formulation requires the limit $\varepsilon \downarrow 0$. The evolution equations of the local values of the conserved quantities in the limit are called the Euler equations of the system.

In our model the corresponding situation is the following. Since we only deal with deterministic configurations, the appropriate notion of equilibrium for these systems is stationarity of the solution, i.e., velocity 0 of all particles. Lang calls them rigid states. For the relation to other notions of equilibrium for random configurations see Lang[2] and Zessin.[3]

As in the case of the customary gradient system, there is only one non-trivial conserved quantity, namely the particle number. The corresponding equilibrium distributions are parametrized by the density and essentially characterized by equidistance of the particles. In addition, there is a trivial conserved quantity, namely the velocity, whose total sum is 0.

Thus, we study their local distributions, given by the measures

$$\rho_t^\varepsilon = \varepsilon \sum_i \delta_{q_i(t)} \tag{1.2a}$$

$$v_t^\varepsilon = \varepsilon \sum_i v_i(t)\, \delta_{q_i(t)} \qquad \text{with} \quad v_i = \frac{dq_i}{dt} \tag{1.2b}$$

For a sufficiently smooth test function φ we get

$$\frac{d}{dt} \int \varphi \, d\rho_t^\varepsilon = \frac{d}{dt}\, \varepsilon \sum_i \varphi(q_i) = \varepsilon \sum_i \varphi'(q_i) \cdot v_i = \int \varphi'(q)\, dv_t^\varepsilon \tag{1.3}$$

i.e., $(d/dt)\,\rho_t^\varepsilon = -(\partial/\partial q) v_t^\varepsilon$ holds in a weak sense.

In the next section we prove sequential compactness for these measures. The procedure in the next and the beginning of Section 3 is the same as in Ref. 7 with modifications of the estimates. In Section 3 we study the properties of limit distributions as $\varepsilon \downarrow 0$. We prove the existence of a density of these distributions and the validity of local equilibrium. It follows that the limit velocity distribution is a functional of the limit density distribution and Eq. (1.3) becomes in the limit the nonlinear diffusion equation

$$\frac{\partial}{\partial t} \rho_t(q) = \frac{\partial^2}{\partial q^2}\left(\mathbf{F}\left(\frac{1}{\rho_t(q)}\right)\right) \tag{1.4}$$

We give a formal argument for the validity of this equation. For this purpose we transform (1.3):

$$\begin{aligned}
\frac{d}{dt}\int \varphi \, d\rho_t^\varepsilon &= \varepsilon \sum_i \varphi'(q_i)\cdot v_i \\
&= \varepsilon \sum_i \varphi'(q_i) \sum_{j:\,|j-i|=1} \varepsilon^{-1}\mathbf{F}\left(\frac{q_i-q_j}{\varepsilon}\right) \\
&= \frac{\varepsilon}{2} \sum_{(i,j)\in\mathcal{N}} \frac{\varphi'(q_i)-\varphi'(q_j)}{\varepsilon}\mathbf{F}\left(\frac{q_i-q_j}{\varepsilon}\right), \\
&\qquad \text{with} \quad \mathcal{N}=\{(i,j)\colon |i-j|=1\} \\
&\approx \frac{\varepsilon}{2}\sum_i \varphi''(q_i) \sum_{j:\,|j-i|=1} \frac{q_i-q_j}{\varepsilon}\mathbf{F}\left(\frac{q_i-q_j}{\varepsilon}\right)
\end{aligned}$$

Now assume that local equilibrium holds.

Then by equidistance $(q_{i+1}-q_i)/\varepsilon$ and $(q_i-q_{i-1})/\varepsilon$ are approximately $1/\rho_t(q_i)$ and we get as $\varepsilon \downarrow 0$

$$\frac{d}{dt}\int \varphi \, d\rho_t = \int \varphi'' \frac{1}{\rho_t}\mathbf{F}\left(\frac{1}{\rho_t}\right) d\rho_t = \int \varphi''(q)\,\mathbf{F}\left(\frac{1}{\rho_t(q)}\right) dq$$

We shall even derive Eq. (1.4) in a stronger sense, indicating the validity of local equilibrium of a higher order.

In the final section uniqueness is proved for the initial value problem of Eq. (1.4). As a consequence, we get in the hydrodynamic limit the convergence of the density distribution for all times, if it converges at time 0, and the validity of the Euler equation (1.4) for the limit density distribution in a suitable weak sense.

2. COMPACTNESS

We first state the general assumptions, which will be made throughout, without further mention.

The potential

$$\mathbf{\Phi}: \mathbf{R}\backslash\{0\} \to \mathbf{R}^+$$

is a twice continuously differentiable function with the following properties:

1. Symmetry: $\Phi(q) = \Phi(-q)$ for $q \neq 0$.
2. Convexity: There exists $0 < R \leqslant +\infty$, such that Φ is strictly convex on $(0, R]$ and identically 0 on $[R, +\infty)$, if $R < \infty$, resp. decreases to 0 as $q \to \infty$, if $R = +\infty$.
3. Behavior near 0: (a) $\Phi(q) \to +\infty$ as $|q| \to 0$; (b) there exists $\alpha, \beta > 0$ such that, for $|q|$ sufficiently small,

$$|q \cdot \Phi'(q)| \leqslant \alpha\Phi(q)$$

$$|q \cdot \Phi''(q)| \leqslant \beta\, |\Phi'(q)|$$

The last condition prevents singularities of infinite order at 0. Then easily follows:

3′. (b) There exists $\gamma, \delta > 0$ such that

$$|q|^2 \cdot |\Phi''(q)| \leqslant \gamma\Phi(q) + \delta \qquad \text{for every} \quad q \neq 0$$

In fact, we shall only need 3′b instead of 3b.

These conditions are similar to those of Lang.[2]

Concerning the initial configurations we assume the following: For each $\varepsilon > 0$ there is a finite configuration of particles

$$\{q_i^\varepsilon(0);\, 1 \leqslant i \leqslant N^\varepsilon\} \subset \mathbf{R}$$

with $q_i^\varepsilon(0) < q_{i+1}^\varepsilon(0)$ for $1 \leqslant i \leqslant N^\varepsilon - 1$, such that the suitably normalized particle number (mass) and energy are uniformly bounded in $\varepsilon > 0$:

$$\varepsilon \cdot N^\varepsilon \leqslant M \tag{2.1}$$

$$\frac{\varepsilon}{2} \sum_{(i,j) \in \mathcal{N}^\varepsilon} \Phi\left(\frac{q_i^\varepsilon(0) - q_j^\varepsilon(0)}{\varepsilon}\right) \leqslant E \tag{2.2}$$

with the system of neighbored indices

$$\mathcal{N}^\varepsilon := \{(i, j)\colon 1 \leqslant i, j \leqslant N^\varepsilon;\, |i - j| = 1\}$$

We study the time evolution of the configurations $\{q_i^\varepsilon(t);\, 1 \leqslant i \leqslant N^\varepsilon\}$ for $t \geqslant 0$, given by Eq. (1.1) with $\mathbf{F} = -\mathbf{\Phi}'$.

Obviously, the order of the particles is preserved in time.

To make the formulas more lucid, we omit the explicit quotation of ε in the configurations, if this causes no confusion, keeping it in ρ_t^ε and v_t^ε [see (1.2)].

First we show that the assumptions on the initial configurations are preserved in time, which is only nontrivial for the boundedness of the energy, and that $\varepsilon \sum_i v_i(t)^2$ with $v_i = dq_i/dt$ is uniformly bounded in $\varepsilon > 0$ and t uniformly distant from 0.

We shall make frequent use of the first and second derivatives of the energy, which are deduced as in the customary case,[2]

$$\frac{d}{dt} \frac{\varepsilon}{2} \sum_{(i,j) \in \mathcal{N}^\varepsilon} \Phi\left(\frac{q_i(t) - q_j(t)}{\varepsilon}\right) = -\varepsilon \sum_i v_i(t)^2 \leqslant 0 \tag{2.3}$$

$$\frac{d}{dt} \varepsilon \sum_i v_i(t)^2 = \varepsilon \sum_{(i,j) \in \mathcal{N}^\varepsilon} \varepsilon^{-2} [v_i(t) - v_j(t)]^2 \,\mathrm{F}'\left(\frac{q_i(t) - q_j(t)}{\varepsilon}\right) \leqslant 0 \tag{2.4}$$

Lemma 2.1.

(i) $$\frac{\varepsilon}{2} \sum_{(i,j) \in \mathcal{N}^\varepsilon} \Phi\left(\frac{q_i(t) - q_j(t)}{\varepsilon}\right) \leqslant E \qquad \text{for} \quad \varepsilon > 0, \quad t \geqslant 0$$

(ii) $$\int_0^\infty \varepsilon \sum_i v_i(t)^2 \, dt \leqslant E \qquad \text{for} \quad \varepsilon > 0$$

$$\varepsilon \sum_i v_i(t)^2 \leqslant \frac{E}{t} \qquad \text{for} \quad \varepsilon > 0, \quad t > 0$$

Proof. Part (i) and the first estimate of (ii) are an easy consequence of (2.3).

The latter and the monotonicity of $\varepsilon \sum_i v_i(t)^2$ imply the second estimate of (ii):

$$E \geqslant \int_0^t \varepsilon \sum_i v_i(s)^2 \, ds \geqslant \varepsilon \sum_i v_i(t)^2 \cdot t$$

The main object of our interest is the behavior of the time evolution of the local density distribution represented by the measures

$$\rho_t^\varepsilon = \varepsilon \sum_i \delta_{q_i(t)}$$

in (1.2a) as $\varepsilon \downarrow 0$. For this purpose we also have to study the corresponding behavior of the local velocity distribution represented by the signed measures

$$\nu_t^\varepsilon = \varepsilon \sum_i v_i(t)\, \delta_{q_i(t)}$$

in (1.2b). In the sequel, we denote for convenience by measures those with real values, as we tacitly did it in the introduction.

In this section we prove convergence of subsequences for these measures as weakly continuous functions of time as $\varepsilon \downarrow 0$.

Proposition 2.2. If $\{\rho_0^\varepsilon; \varepsilon > 0\}$ is tight, then for $0 < t_0 < T$ the sets of measures $\{\rho_t^\varepsilon; \varepsilon > 0, 0 \leqslant t \leqslant T\}$ and $\{\nu_t^\varepsilon; \varepsilon > 0, t_0 \leqslant t \leqslant T\}$ are tight.

Proof. By Lemma 2.1 there holds, for $C > 0$, $t \geqslant 0$,

$$\begin{aligned}\varepsilon \sum_i \mathbf{1}_{\{|q_i(t) - q_i(0)| \geqslant C\}} &\leqslant \frac{1}{C^2}\, \varepsilon \sum_i |q_i(t) - q_i(0)|^2 \\ &= \frac{1}{C^2}\, \varepsilon \sum_i \left| \int_0^t v_i(s)\, ds \right|^2 \\ &\leqslant \frac{1}{C^2}\, \varepsilon \sum_i t \int_0^t v_i(s)^2\, ds \leqslant \frac{Et}{C^2}\end{aligned}$$

which implies the tightness of the density distribution, if it holds for $t = 0$.

The tightness of the velocity distribution follows from

$$\begin{aligned}&\varepsilon \sum_i |v_i(t)|\, \mathbf{1}_{\{|q_i(t)| \geqslant C\}} \\ &\leqslant \left[\varepsilon \sum_i v_i(t)^2 \right]^{1/2} \cdot \left(\varepsilon \sum_i \mathbf{1}_{\{|q_i(t)| \geqslant C\}} \right)^{1/2} \\ &\leqslant \left(\frac{E}{t} \right)^{1/2} \left(\varepsilon \sum_i \mathbf{1}_{\{|q_i(t)| \geqslant C\}} \right)^{1/2}\end{aligned}$$

Theorem 2.3. Let $\{\rho_0^\varepsilon; \varepsilon > 0\}$ be tight. Then for each sequence $\varepsilon_n \downarrow 0$ there exists a subsequence $\varepsilon_{n(k)} \downarrow 0$ such that $\rho_t^{\varepsilon_{n(k)}}$ converges weakly for each $t \geqslant 0$ and $\nu_t^{\varepsilon_{n(k)}}$ converges weakly for each $t > 0$. The limit measures are weakly continuous in t.

Proof. It suffices to prove the result for $t \in [0, T]$, resp. $(0, T]$, for each $T > 0$.

Let $T > 0$ be fixed and $D \subset (0, T]$ be a denumerable dense subset and $D_0 = D \cup \{0\}$. Proposition 2.2 and a diagonal procedure yield for a sequence $\varepsilon_n \downarrow 0$ the existence of a subsequence $\varepsilon_{n(k)} \downarrow 0$ such that $\rho_t^{\varepsilon_{n(k)}}$ and $\nu_t^{\varepsilon_{n(k)}}$ converge weakly for $t \in D_0$, resp. $t \in D$.

We show that this subsequence satisfies the assertions of the theorem. For that purpose we prove the uniform continuity of $\int \varphi \, d\rho_t^\varepsilon$ and $\int \varphi \, d\nu_t^\varepsilon$ for $\varepsilon > 0$ and $t \in [0, T]$, resp. $t \in [t_0, T]$, for $0 < t_0 < T$ with fixed $\varphi \in C_b^1$, the set of bounded, continuously differentiable functions with bounded derivative.

Then both the convergence of the subsequence for all times and the weak continuity of the limit distributions easily follow in connection with tightness, since every corresponding limit is uniquely determined.

We shall frequently use the uniform norm and hence denote it without mark:

$$\|\varphi\| := \sup\{|\varphi(q)| : q \in \mathbf{R}\}$$

Now let $\varphi \in C_b^1$. Then there holds

$$\begin{aligned} \left| \frac{d}{dt} \int \varphi \, d\rho_t^\varepsilon \right| &= \left| \varepsilon \sum_i \varphi'(q_i(t)) \, v_i(t) \right| \\ &\leqslant \left[\varepsilon \sum_i \varphi'(q_i(t))^2 \right]^{1/2} \left[\varepsilon \sum_i v_i(t)^2 \right]^{1/2} \\ &\leqslant (ME)^{1/2} \, \|\varphi'\| \, t^{-1/2} \end{aligned}$$

from which the assertion follows for the density distribution.

The case of the velocity distribution is more difficult. We have

$$\begin{aligned} \frac{d}{dt} \int \varphi \, d\nu_t^\varepsilon &= \varepsilon \sum_i \varphi'(q_i(t)) \, v_i(t)^2 + \varepsilon \sum_i \varphi(q_i(t)) \cdot \frac{dv_i}{dt} \\ &= \varepsilon \sum_i \varphi'(q_i(t)) \, v_i(t)^2 \\ &\quad + \varepsilon \sum_i \varphi(q_i(t)) \sum_{j: |j-i| = 1} \varepsilon^{-2} \mathbf{F}' \left(\frac{q_i(t) - q_j(t)}{\varepsilon} \right) [v_i(t) - v_j(t)] \\ &= \varepsilon \sum_i \varphi'(q_i(t)) \, v_i(t)^2 \\ &\quad + \frac{\varepsilon}{2} \sum_{(i,j) \in \mathcal{N}^\varepsilon} \frac{\varphi(q_i(t)) - \varphi(q_j(t))}{\varepsilon} \mathbf{F}' \left(\frac{q_i(t) - q_j(t)}{\varepsilon} \right) \frac{v_i(t) - v_j(t)}{\varepsilon} \end{aligned}$$

The first term is easy to handle. For the second term we need property 3′b of the potential Φ and (2.4):

$$
\begin{aligned}
&\left|\varepsilon \sum_i \varphi(q_i(t)) \frac{dv_i}{dt}\right| \\
&\quad \leqslant \frac{1}{2}\|\varphi'\| \left[\varepsilon \sum_{(i,j)\in\mathcal{N}^\varepsilon} \left(\frac{q_i(t)-q_j(t)}{\varepsilon}\right)^2 \left|\mathbf{F}'\left(\frac{q_i(t)-q_j(t)}{\varepsilon}\right)\right|\right]^{1/2} \\
&\quad\quad \times \left[\varepsilon \sum_{(i,j)\in\mathcal{N}^\varepsilon} \left|\mathbf{F}'\left(\frac{q_i(t)-q_j(t)}{\varepsilon}\right)\right| \left(\frac{v_i(t)-v_j(t)}{\varepsilon}\right)^2\right]^{1/2} \\
&\quad \leqslant \frac{1}{2}\|\varphi'\| \left\{\varepsilon \sum_{(i,j)\in\mathcal{N}^\varepsilon} \left[\gamma\Phi\left(\frac{q_i(t)-q_j(t)}{\varepsilon}\right)+\delta\right]\right\}^{1/2} \left[-\frac{d}{dt}\varepsilon\sum_i v_i(t)^2\right]^{1/2} \\
&\quad \leqslant C\,\|\varphi'\| \left[-\frac{d}{dt}\varepsilon\sum_i v_i(t)^2\right]^{1/2}
\end{aligned}
$$

with $C=\frac{1}{2}(\gamma E+2\delta M)^{1/2}$.

There follows for $\varepsilon>0$ and $0<t<t+h$:

$$
\begin{aligned}
&\left|\int \varphi\, dv_{t+h}^\varepsilon - \int \varphi\, dv_t^\varepsilon\right| \\
&\quad \leqslant \|\varphi'\| \left(\int_t^{t+h} \left\{\left[\varepsilon\sum_i v_i(s)^2\right] + C\left[-\frac{d}{ds}\varepsilon\sum_i v_i(s)^2\right]^{1/2}\right\} ds\right) \\
&\quad \leqslant \|\varphi'\| \left(\frac{E}{t}h + Ch^{1/2}\left\{\int_t^{t+h}\left[-\frac{d}{ds}\varepsilon\sum_i v_i(s)^2\right] ds\right\}^{1/2}\right) \\
&\quad \leqslant \|\varphi'\| \left[\frac{E}{t}h + C\left(\frac{E}{t}h\right)^{1/2}\right]
\end{aligned}
$$

which finally yields the assertion for the velocity distribution.

Remark. The proof even shows the uniform convergence on compact time intervals with respect to the dual of the bounded Lipschitz norm[8]:

$$
\|\mu\|_{\mathrm{BL}}^* := \sup\left\{\left|\int \varphi\, d\mu\right| : \|\varphi\|_{\mathrm{BL}} \leqslant 1\right\}
$$

with

$$
\|\varphi\|_{\mathrm{BL}} = \|\varphi\| + \sup\left\{\frac{|\varphi(q_1)-\varphi(q_2)|}{|q_1-q_2|} : q_1, q_2 \in \mathbf{R},\ q_1 \neq q_2\right\}
$$

This norm generates the weak convergence for nonnegative measures, and so uniform convergence holds for the density distribution.

3. THE LIMIT DYNAMICS

In this section we investigate the behavior of limit density and velocity distributions.

Theorem 3.1. Let $\varepsilon_n \downarrow 0$ with $\rho_t^{\varepsilon_n} \to \rho_t$ and $\nu_t^{\varepsilon_n} \to \nu_t$ weakly for $t \geqslant 0$, resp. $t > 0$. Then ρ_t is absolutely continuous with respect to the Lebesgue measure for $t \geqslant 0$ and ν_t is absolutely continuous with respect to ρ_t for $t > 0$.

We prove the absolute continuity of ρ_t with an energy estimate of a variational principle.

Lemma 3.2. Let $a < b$ and $N \in \mathbf{N}$. Then for $a \leqslant x_1 < \cdots < x_{N+1} \leqslant b$

$$\sum_{i=1}^{N} \Phi(x_{i+1} - x_i) \geqslant N\Phi\left(\frac{b-a}{N}\right)$$

Equality holds in the case $(b-a)/N \leqslant R$, iff $x_{i+1} - x_1 = (b-a)/N$ for $1 \leqslant i \leqslant N$, and in the case $(b-a)/N > R$, iff $x_{i+1} - x_i \geqslant R$ for $1 \leqslant i \leqslant N$.

Proof. By the convexity of Φ there follows for fixed $x_1 < X_{N+1}$

$$\sum_{i=1}^{N} \Phi(x_{i+1} - x_i) \geqslant N\Phi\left(\frac{x_{N+1} - x_1}{N}\right)$$

and the right side is minimal for maximal $x_{N+1} - x_1$, i.e., for $x_1 = a$, $x_{N+1} = b$.

The assertion concerning the equality is a consequence of the strict convexity.

Proof of Theorem 3.1. For simplicity we omit the index n for assertions that hold for every $\varepsilon > 0$.

Let $t \geqslant 0$ be fixed and $I \subset \mathbf{R}$ be an interval with length $|I|$ such that $\rho_t(\partial I) = 0$. We denote by $N_t^\varepsilon(I) = \#\{i: q_i^\varepsilon(t) \in I\}$. Since Φ is nonnegative, there follows from Lemma 3.1

$$E \geqslant \varepsilon \sum_{i:\, q_i, q_{i+1} \in I} \Phi\left(\frac{q_{i+1} - q_i}{\varepsilon}\right) \geqslant \varepsilon[N_t^\varepsilon(I) - 1]\Phi\left(\frac{\varepsilon^{-1}|I|}{N_t^\varepsilon(I) - 1}\right)$$

With $\varepsilon_n N_t^{\varepsilon_n}(I) \to \rho_t(I)$ this yields

$$\rho_t(I)\Phi\left(\frac{|I|}{\rho_t(I)}\right) \leqslant E$$

which can easily be extended to hold for every interval $I \subset \mathbf{R}$ by approximation.

Now let $I_1,..., I_m$ be disjoint intervals. Then there follows by a similar argument as in the proof of Lemma 3.2

$$\begin{aligned} E &\geqslant \varepsilon \sum_{j=1}^{m} \sum_{i:\, q_i, q_{i+1} \in I_j} \Phi\left(\frac{q_{i+1}-q_i}{\varepsilon}\right) \\ &\geqslant \sum_{j=1}^{m} \rho_t(I_j)\Phi\left(\frac{|I_j|}{\rho_t(I_j)}\right) \\ &\geqslant \left[\sum_{j=1}^{m} \rho_t(I_j)\right] \Phi\left(\frac{\sum_{j=1}^{m} |I_j|}{\sum_{j=1}^{m} \rho_t(I_j)}\right) \end{aligned}$$

The function $\rho\Phi(\tau/\rho)$ is increasing in $\rho \geqslant 0$ and decreasing in $\tau \geqslant 0$ with $\rho\Phi(\tau/\rho) \to \infty$ as $\tau \to 0$ with $\rho > 0$ fixed.

Hence for each $\eta > 0$ there exists $\delta > 0$ such that

$$\sum_{j=1}^{m} \rho_t(I_j) \leqslant \eta \qquad \text{for} \quad \sum_{j=1}^{m} |I_j| \leqslant \delta$$

The absolute continuity of v_t with respect to ρ_t follows from the estimate

$$\varepsilon \sum_i \mathbf{1}_C(q_i(t))\, |v_i(t)| \leqslant \left[\varepsilon \sum_i \mathbf{1}_C(q_i(t))\right]^{1/2} \left[\varepsilon \sum_i v_i(t)^2\right]^{1/2}$$

We shall identify the measure ρ_t with its density and denote the density of v_t with respect to ρ_t by u_t.

Theorem 3.3. Under the assumptions of Theorem 3.1 for each $t > 0$ the function $\mathbf{F}(1/\rho_t)$ is absolutely continuous—and in that way uniquely defined—and the relation

$$\frac{\partial}{\partial q}\left(\mathbf{F}\left(\frac{1}{\rho_t(q)}\right)\right) = -u_t(q)\, \rho_t(q)$$

holds for a.e. q with respect to the Lebesgue measure.

Corollary 3.4. Under the assumptions of Theorem 3.1 the limit density distribution ρ_t $(t \geqslant 0)$ satisfies the equation

$$\frac{\partial}{\partial t}\rho_t(q) = \frac{\partial^2}{\partial q^2}\left(\mathbf{F}\left(\frac{1}{\rho_t(q)}\right)\right)$$

in the weak sense

$$\frac{d}{dt}\int \varphi(q)\,\rho_t(q)\,dq = -\int \varphi'(q)\frac{\partial}{\partial q}\left(\mathbf{F}\left(\frac{1}{\rho_t(q)}\right)\right)dq \qquad \text{for} \quad \varphi \in C_b^1;\ t>0$$

Proof. For $a<b$ there holds

$$\begin{aligned} v_t^\varepsilon([a,b)) &= \varepsilon \sum_i \mathbf{1}_{[a,b)}(q_i(t))\, v_i(t) \\ &= \varepsilon \sum_i \mathbf{1}_{[a,b)}(q_i(t))\, \varepsilon^{-1}\left[-\mathbf{F}\left(\frac{q_{i+1}(t)-q_i(t)}{\varepsilon}\right)+\mathbf{F}\left(\frac{q_i(t)-q_{i-1}(t)}{\varepsilon}\right)\right] \end{aligned}$$

For $\varepsilon>0$, $t>0$ we define the function

$$r_t^\varepsilon(q) = \frac{q_{i+1}^\varepsilon(t)-q_i^\varepsilon(t)}{\varepsilon} \qquad \text{with} \quad q_i^\varepsilon(t)<q\leqslant q_{i+1}^\varepsilon(t); \quad q\in\mathbf{R}$$

setting $r_t^\varepsilon(q)=\infty$ if no such i exists.

Then we have

$$v_t^\varepsilon([a,b)) = -\mathbf{F}(r_t^\varepsilon(b))+\mathbf{F}(r_t^\varepsilon(a)) \qquad \text{for} \quad a<b \tag{3.1}$$

with the convention $\mathbf{F}(+\infty)=0$.

Since $v_t^{\varepsilon_n}\to v_t$ weakly, there follows for $a=-\infty$, $b=q$, that there exists

$$\bar{\mathbf{F}}_t(q) := \lim_{n\to\infty} \mathbf{F}(r_t^{\varepsilon_n}(q)) = -v_t((-\infty,q)) \tag{3.2}$$

and by Theorem 3.1 there holds

$$\bar{\mathbf{F}}_t(b)-\bar{\mathbf{F}}_t(a) = -\int_a^b u_t(q)\,\rho_t(q)\,dq \qquad \text{for} \quad a<b$$

Hence, $\bar{\mathbf{F}}_t$ is absolutely continuous with

$$\frac{\partial}{\partial q}\bar{\mathbf{F}}_t(q) = -u_t(q)\,\rho_t(q)$$

for a.e. q with respect to the Lebesgue measure.

It remains to show

$$\bar{\mathbf{F}}_t(q) = \mathbf{F}\left(\frac{1}{\rho_t(q)}\right); \qquad t>0, \quad q\in\mathbf{R} \tag{3.3}$$

For this purpose we need the validity of local equilibrium, which is of interest in itself.

Proposition 3.5. Let $t>0$. In the domain $\{q\colon \mathbf{F}(1/[\rho_t(q)])>0\}$ the function ρ_t is absolutely continuous, too, and for each $q\in\mathbf{R}$ with $\mathbf{F}(1/[\rho_t(q)])>0$ and $\eta>0$ there exists $\varepsilon_0>0$, $\delta>0$ with

$$\left|\frac{q_{i+1}^{\varepsilon_n}(t)-q_i^{\varepsilon_n}(t)}{\varepsilon}-\frac{1}{\rho_t(q)}\right|\leqslant\eta \qquad \text{for} \quad 0<\varepsilon_n<\varepsilon_0, \quad |q_i^{\varepsilon_n}(t)-q|\leqslant\delta$$

For each $q\in\mathbf{R}$ with $\mathbf{F}(1/[\rho_t(q)])=0$ and $\eta>0$ there exists $\varepsilon_0>0$, $\delta>0$ with

$$\frac{q_{i+1}^{\varepsilon_n}(t)-q_i^{\varepsilon_n}(t)}{\varepsilon}\geqslant R-\eta \qquad \text{for} \quad 0<\varepsilon_n<\varepsilon_0, \quad |q_i^{\varepsilon_n}(t)-q|\leqslant\delta$$

Proof. We first prove this proposition for $\bar{\mathbf{F}}_t$ instead of $\mathbf{F}(1/\rho_t)$ and then show (3.3). The crucial estimate is the following: For $a\leqslant q'<q''\leqslant b$ there holds

$$\begin{aligned}
&|\mathbf{F}(r_t^{\varepsilon}(q''))-\mathbf{F}(r_t^{\varepsilon}(q'))|\\
&\quad\leqslant\left[\varepsilon\sum_i \mathbf{1}_{[q',q'')}(q_i(t))\right]^{1/2}\left[\varepsilon\sum_i v_i(t)^2\right]^{1/2}\\
&\quad\leqslant\left[\varepsilon\sum_i \mathbf{1}_{[a,b)}(q_i(t))\right]^{1/2}\left(\frac{E}{t}\right)^{1/2}
\end{aligned}\tag{3.4}$$

Now let $q\in\mathbf{R}$ be fixed. Since

$$\varepsilon_n\sum_i \mathbf{1}_{[a,b)}(q_i(t))\to\rho_t([a,b)) \qquad \text{as} \quad \varepsilon_n\downarrow 0$$

and

$$\rho_t([q-\delta,q+\delta))\to 0 \qquad \text{as} \quad \delta\downarrow 0$$

there follows from (3.4): For each $\eta>0$ there exists $\delta>0$, $\varepsilon_0>0$ with

$$|\mathbf{F}(r_t^{\varepsilon_n}(q'))-\mathbf{F}(r_t^{\varepsilon_n}(q))|\leqslant\eta \qquad \text{for} \quad 0<\varepsilon_n<\varepsilon_0, \quad |q'-q|\leqslant\delta \tag{3.5}$$

First we consider the case $\bar{\mathbf{F}}_t(q)>0$. Then by (3.2) and (3.5) there exist $\varepsilon'>0$, $\delta'>0$ with

$$\mathbf{F}(r_t^{\varepsilon_n}(q'))\geqslant\tfrac{1}{2}\bar{\mathbf{F}}_t(q')>0 \qquad \text{for} \quad 0<\varepsilon_n<\varepsilon_0, \quad |q'-q|\leqslant\delta$$

Under these conditions we can invert the function $\mathbf{F}$ with the inverse function having bounded derivative. Again by (3.2) and (3.5) there follows:

$$r_t^{\varepsilon_n}(q') \to r_t(q') = \mathbf{F}^{-1}(\bar{\mathbf{F}}(r_t(q'))) \qquad \text{as} \quad \varepsilon_n \downarrow 0$$

and r_t is absolutely continuous for $|q'-q| \leqslant \delta'$.

Furthermore, for each $\eta > 0$ there exist $\varepsilon_0 > 0$, $\delta > 0$ with

$$\begin{aligned} |r_t^{\varepsilon_n}(q') - r_t^{\varepsilon_n}(q)| &\leqslant \eta/2 \qquad \text{for} \quad 0 < \varepsilon_n < \varepsilon_0, \quad |q'-q| \leqslant \delta \\ |r_t^{\varepsilon_n}(q) - r_t(q)| &\leqslant \eta/2 \qquad \text{for} \quad 0 < \varepsilon_n < \varepsilon_0 \end{aligned} \tag{3.6}$$

We get a lower, resp. upper, bound of the number of particles in $[a, b)$ by estimating the distance of neighbored particles. We have

$$\rho_t^{\varepsilon_n}([a,b)) = \varepsilon_n \sum_i \mathbf{1}_{[a,b)}(q_i(t)) \geqslant \left(\sup\left\{\frac{q_{i+1}-q_i}{\varepsilon_n} : a \leqslant q_i < b\right\}\right)^{-1} (b-a)$$

Now fix $\eta > 0$ and let $\delta > 0$ as in (3.6). There easily follows with $\varepsilon_n \downarrow 0$

$$\rho_t(q') \geqslant [r_t(q') + \eta]^{-1} \qquad \text{for} \quad |q'-q| \leqslant \delta$$

and similarly

$$\rho_t(q') \leqslant [r_t(q') - \eta]^{-1} \qquad \text{for} \quad |q'-q| \leqslant \delta$$

Consequently,

$$\rho_t(q) = [r_t(q)]^{-1}$$

$$\bar{\mathbf{F}}_t(q) = \mathbf{F}\left(\frac{1}{\rho_t(q)}\right)$$

which finishes the proof of this case in combination with (3.6).

In the case $\bar{\mathbf{F}}_t(q) = 0$ we cannot invert $\mathbf{F}$, and $\rho_t(q)$ may not be uniquely defined, but similarly to the preceding reasoning we get the following: For each $\eta > 0$ there exist $\varepsilon_0 > 0$, $\delta > 0$ with

$$\begin{aligned} r_t^{\varepsilon_n}(q') &\geqslant R - \eta \qquad \text{for} \quad 0 < \varepsilon_n < \varepsilon_0, \quad |q'-q| \leqslant \delta \\ \rho_t(q') &\leqslant (R-\eta)^{-1} \qquad \text{for} \quad \text{a.e. } q' \text{ with } |q'-q| \leqslant \delta \end{aligned}$$

Though ρ_t may not be uniquely defined, we get

$$\bar{\mathbf{F}}_t(q) = \mathbf{F}\left(\frac{1}{\rho_t(q)}\right) = 0 \qquad \text{on the set} \quad \{q : \bar{\mathbf{F}}_t(q) = 0\}$$

If we choose such a version of ρ_t, that $\mathbf{F}(1/\rho_t)$ is absolutely continuous.

Proof of Corollary 3.4. We start from the integrated version of (1.3):

$$\int \varphi \, d\rho_t^{\varepsilon_n} = \int \varphi \, d\rho_0^{\varepsilon_0} + \int_0^t \left(\int \varphi' \, dv_s^{\varepsilon_n} \right) ds; \qquad \varphi \in C_b^1, \quad t \geqslant 0$$

The estimate (2.5) permits us to interchange the limit $v_s^{\varepsilon_n} \to v_s$ $(0 < s \leqslant t)$ with the integration. Finally, we differentiate with respect to time and get the desired result.

Remark. The fact that Eq. (1.4) holds in the sense stated in Corollary 3.4 compared with the weaker version formally derived in the introduction suggests that within the range of the potential not only does the distance of the particles become locally constant in the limit, but that this also holds for the properly rescaled change of these distances, which means local equilibrium of a higher order.

4. UNIQUENESS

We are now ready to prove our main result.

Theorem 4.1. Let $\rho_0^\varepsilon \to \rho_0$ as $\varepsilon \downarrow 0$ with bounded density ρ_0. Then for each $t \geqslant 0$ there exists ρ_t with $\rho_t^\varepsilon \to \rho_t$ weakly as $\varepsilon \downarrow 0$. Now, ρ_t is weakly continuous in $t \geqslant 0$, has a density with respect to the Lebesgue measure, and satisfies the differential equation

$$\frac{\partial \rho_t(q)}{\partial t} = \frac{\partial^2}{\partial q^2} \left(\mathbf{F} \left(\frac{1}{\rho_t(q)} \right) \right)$$

in the weak sense

$$\frac{d}{dt} \int \varphi(q) \, \rho_t(q) \, dq = -\int \frac{\partial}{\partial q} \left(\mathbf{F} \left(\frac{1}{\rho_t(q)} \right) \right) \cdot \varphi'(q) \, dq \qquad \text{for} \quad \varphi \in C_b^1, \ t > 0$$

After the results of the preceding sections it remains to prove that every limit ρ_t, v_t ($t \geqslant 0$, resp. $t > 0$) of a convergent sequence $\rho_t^{\varepsilon_n}$, $v_t^{\varepsilon_n}$ is uniquely determined as a solution of an initial value problem of the differential equation in the stated version.

We use the methods of Alt and Luckhaus,[9] who study similar differential equations. Their results, however, are not directly applicable, since their primary function corresponds to $(\partial/\partial q)\{\mathbf{F}([1/\rho_t(q)])\}$, which is not invertible if $R < \infty$. But their methods can be transferred almost literally. The main task is to verify that the assumptions of Ref. 9 hold in our case. For that purpose we have to extend the validity of the differential equation to appropriate functions, which also depend on time.

We use the customary denotation of $H^{n,p}$ ($n \geqslant 0$ integer, $p \geqslant 1$ real) with specified domain as the Sobolev space of functions with generalized derivative up to order n in L^p. All L^p and $H^{n,p}$ spaces refer to the Lebesgue measure.

$\mathscr{M} = \mathscr{M}(\mathbf{R})$ is the space of finite Borel measures on $\mathbf{R}$ with the total variation norm $\|\mu\|_v := |\mu|\ (\mathbf{R})$. The space $\mathscr{M}$ is the dual of $C_0(\mathbf{R})$, the set of continuous functions on $\mathbf{R}$, which vanish at infinity, with the uniform norm $\|\cdot\|$ (see Section 2).

We denote the dual of a Banach space E by E^*.

Now let $\varepsilon_n \downarrow 0$ with $\rho_t^{\varepsilon_n} \to \rho_t$ and $v_t^{\varepsilon_n} \to v_t$ weakly for $t \geqslant 0$, resp. $t > 0$.

For fixed $T > 0$ we conceive $\partial \rho_t / \partial t$ as a functional, defined by

$$\int_0^T \left\langle \varphi_t, \frac{\partial \rho_t}{\partial t} \right\rangle dt := \lim_{h \downarrow 0} \int_0^T \left[\int \varphi_t(q) \frac{\rho_{t+h}(q) - \rho_t(q)}{h}\, dq \right] dt$$

on spaces of suitable test functions φ for which this limit exists.

Lemma 4.2. Let $T > 0$. Then:

(i) $\rho_{\cdot} \in L^\infty([0, T]; L^1(\mathbf{R}))$

and

$$\int_0^T \left\langle \varphi_t, \frac{\partial \rho_t}{\partial t} \right\rangle dt = -\int_0^T \left\{ \int \frac{\partial \varphi_t(q)}{\partial q} [\rho_t(q) - \rho_0(q)]\, dq \right\} dt$$

holds for

$$\varphi \in L^2([0, T]; H^{1,2}(\mathbf{R})) \cap H^{1,1}([0, T]; L^\infty(\mathbf{R})) \qquad \text{with} \quad \varphi_T \equiv 0$$

(ii) $\dfrac{\partial}{\partial q}\left(\mathbf{F}\left(\dfrac{1}{\rho}\right)\right) \in L^2([0, T] \times \mathbf{R})$

(iii) $\dfrac{\partial \rho_t}{\partial t} \in (L^2([0, T]; H^{1,2}(\mathbf{R})))^*$

with

$$\int_0^T \left\langle \varphi_t, \frac{\partial \rho_t}{\partial t} \right\rangle dt = -\int_0^T \left[\int \frac{\partial \varphi_t(q)}{\partial q} \frac{\partial}{\partial q} \left(\mathbf{F}\left(\frac{1}{\rho_t(q)} \right) \right) dq \right] dt$$

for $\varphi \in L^2([0, T]; H^{1,2}(\mathbf{R}))$.

Proof. (i) For $t \geqslant 0$ there holds

$$\|\rho_t\|_{L^1(\mathbf{R})} = \rho_t(\mathbf{R}) \leqslant M$$

and the weak continuity implies the measurability in dependence on t.

The validity of the stated equation follows from this by straightforward manipulations of the integral in the definition of $\partial\rho_t/\partial t$ [see (1.4.1) of Ref. 9].

(ii) We conceive $v_{\cdot}^{\varepsilon_n}$ and $v_{\cdot}$ as elements of $L^2([0, T]; \mathscr{M})$. Then

$$\|v^{\varepsilon_n}\|_{L^2([0,T];\mathscr{M})} = \left\{\int_0^T \left[\varepsilon_n \sum_i |v_i(t)|\right]^2 dt\right\}^{1/2}$$

$$\leqslant \left[\int_0^T M\varepsilon_n \sum_i v_i(t)^2\, dt\right]^{1/2} \leqslant (ME)^{1/2}$$

Since $v_t^{\varepsilon_n} \to v_t$ weakly as $\varepsilon_n \downarrow 0$ for $0 < t \leqslant T$, there follows [see, e.g., Theorem (13.44) of Ref. 10]

$$\int_0^T \left(\int \varphi_t\, dv_t^{\varepsilon_n}\right) dt \to \int_0^T \left(\int \varphi_t\, dv_t\right) dt \qquad \text{as} \quad \varepsilon_n \downarrow 0$$

for $\varphi \in L^2([0, T]; C_0(\mathbf{R}))$.

We need an L^2 estimate for the limit velocity distribution also with respect to the space variable, which we obtain from

$$\left|\int_0^T \left(\int \varphi_t\, dv_t^{\varepsilon_n}\right) dt\right|$$

$$\leqslant \left[\int_0^T \varepsilon_n \sum_i \varphi_t(q_i(t))^2\, dt\right]^{1/2} \left[\int_0^T \varepsilon_n \sum_i v_i(t)^2\, dt\right]^{1/2}$$

$$\leqslant E^{1/2} \left[\int_0^T \varepsilon_n \sum_i \varphi_t(q_i(t))^2\, dt\right]$$

for $\varphi \in L^2([0, T]; C_0(\mathbf{R}))$, which becomes in the limit

$$\left|\int_0^T \left(\int \varphi_t\, dv_t\right) dt\right| \leqslant E^{1/2} \left\{\int_0^T \left[\int \varphi_t(q)^2\, \rho_t(q)\, dq\right] dt\right\}^{1/2}$$

At this point we need the boundedness of ρ_0, which implies the same bound for ρ_t $(t \geqslant 0)$ by the maximum principle applied to Eq. (1.4). Since v_t has the density

$$-\frac{\partial}{\partial q}\left(\mathbf{F}\left(\frac{1}{\rho_t(q)}\right)\right)$$

there exists a constant $C>0$ with

$$\left|\int_0^T \left[\int \varphi_t(q) \frac{\partial}{\partial q}\left(\mathbf{F}\left(\frac{1}{\rho_t(q)}\right)\right) dq\right] dt\right| \leqslant C \left|\int_0^T \left[\int \varphi_t(q)^2 \, dq\right] dt\right|^{1/2}$$

Since $L^2([0, T]; C_0(\mathbf{R}))$ is dense in $L^2([0, T]; L^2(\mathbf{R}))$, there follows

$$\left\| \frac{\partial}{\partial q}\left(\mathbf{F}\left(\frac{1}{\rho_t(q)}\right)\right)\right\|_{L^2([0,T]\times\mathbf{R})} \leqslant C$$

(iii) Let $\varphi \in L^2([0, T]; C_b^1)$. For fixed $t>0$ we apply the differential equation of Corollary 3.4, resp. Theorem 4.1, to φ_t and get

$$\begin{aligned} \lim_{h\downarrow 0} \int \varphi_t(q) \frac{\rho_{t+h}(q)-\rho_t(q)}{h}\, dq &= \frac{d}{ds} \int \varphi_t(q)\, \rho_s(q)\, dq \Big|_{s=t} \\ &= -\int \frac{\partial \varphi_t(q)}{\partial q} \frac{\partial}{\partial q}\left(\mathbf{F}\left(\frac{1}{\rho_t(q)}\right)\right) dq \end{aligned}$$

By the estimates used in part (ii) of the proof one similarly shows that the limit $h\downarrow 0$ can be interchanged with the integration with respect to time, yielding

$$\int_0^T \left\langle \varphi_t, \frac{\partial \rho_t}{\partial t}\right\rangle dt = -\int_0^T \left[\int \frac{\partial \varphi_t(q)}{\partial q}\frac{\partial}{\partial q}\left(\mathbf{F}\left(\frac{1}{\rho_t(q)}\right)\right) dq\right] dt$$

for $\varphi \in L^2([0, T]; C_b^1)$, which can be extended to hold for $\varphi \in L^2([0, T]; H^{1,2}(\mathbf{R}))$. We finally prove the uniqueness of the solution of the differential equation in the version of Lemma 4.2, from which Theorem 4.1 follows, as already mentioned.

Proposition 4.3. There is at most one ρ satisfying the properties of Lemma 4.2 with given ρ_0.

Proof. We proceed as in the proof of Theorem 2.4 of Ref. 9 and only sketch the main steps. Let $\rho_1, \rho_2 \in L^2([0, T]; L^1(\mathbf{R}))$ be two solutions and set $\rho=\rho_1-\rho_2$. Then there exists $\psi \in L^2([0, T]; H^{1,2}(\mathbf{R}))$ with

$$\int_0^T \left[\int \psi_t'(q)\, \varphi_t'(q)\, dq\right] dt = \int_0^T \left[\int \rho_t(q)\, \varphi_t(q)\, dq\right] dt \tag{4.1}$$

for $\varphi \in L^2([0, T]; H^{1,2}(\mathbf{R}))$.

There follows for $h>0$, by the same manipulations of the integral as in Ref. 9, with the use of (4.1),

$$2\int_h^{T+h}\left\langle\frac{\rho_{t-h}-\rho_t}{-h},\psi_t\right\rangle dt+\frac{1}{h}\int_0^h\langle\rho_t,\psi_t\rangle\,dt$$
$$=\frac{1}{h}\int_0^T\left\{\int[\psi'_{t+h}(q)-\psi'_t(q)]^2\,dq\right\}dt+\frac{1}{h}\int_T^{T+h}\left\{\int[\psi'_t(q)]^2\,dq\right\}dt$$

With $h\downarrow 0$ there follows

$$\int_0^T\left\langle\frac{\partial\rho_t}{\partial t},\psi_t\right\rangle dt=\frac{1}{2}\int[\psi'_T(q)]^2\,dq$$

If ψ is inserted into Lemma 4.2(iii), this becomes

$$-\int_0^T\left\{\int\psi'_t(q)\frac{\partial}{\partial q}\left[\mathbf{F}\left(\frac{1}{\rho_{2,t}(q)}\right)-\mathbf{F}\left(\frac{1}{\rho_{1,t}(q)}\right)\right]dq\right\}dt$$
$$=-\int_0^T\left\{\int[\rho_{2,t}(q)-\rho_{1,t}(q)]\left[\mathbf{F}\left(\frac{1}{\rho_{2,t}(q)}\right)-\mathbf{F}\left(\frac{1}{\rho_{1,t}(q)}\right)\right]dq\right\}dt$$

which is $\leqslant 0$ by the monotonicity of $\mathbf{F}$. Since $\frac{1}{2}\int[\psi'_T(q)]^2\,dq\geqslant 0$, both expressions have to be 0. Hence

$$\mathbf{F}\left(\frac{1}{\rho_2}\right)-\mathbf{F}\left(\frac{1}{\rho_1}\right)=0\qquad\text{a.e. on }[0,T]\times\mathbf{R}$$

and thus $\partial\rho_t/\partial t\equiv 0$ on $L^2([0,T];H^{1,2}(\mathbf{R}))$ by Lemma 4.2(iii). From Lemma 4.2(i) we finally obtain $\rho\equiv 0$.

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